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Title:

Analytic One-Dimensional Solutions for the Schwinger Inverse Method

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INTRODUCTION

A novel solution method, based on the Schwinger variational functional, to the inverse transport problem of determining unknown interface locations in a multilayer source/shield system based on gamma-ray flux measurements was recently developed [1]. In one-dimensional (spherical) test problems, the Schwinger inverse method failed to converge when the source/shield and several shield/shield interface locations were unknown, and in other cases it attempted to generate nonphysical models (e.g., with negative radii) [1]. In this paper, the convergence properties of the Schwinger inverse method are analyzed using analytic solutions for a monodirectional slab problem.

ANALYTIC TEST PROBLEM

Consider a one-dimensional homogeneous slab ($0 < r < r_1'$) source of monodirectional gamma-rays (in the positive-r direction) of G discrete energies, and let there be a detector located to the right of the slab at location r_d' such that $r_1' < r_d'$, with a vacuum between the source and the detector. Assuming that any scattered photons lose energy and are removed, the transport equation is

$$\frac{d\psi^g(r)}{dr} + \Sigma_t^g(r)\psi^g(r) = q^g(r), \qquad \psi^g(0) = 0, \quad g = 1,\ldots,G, \tag{1}$$

where the notation is standard and the source $q^g(r)$ has dimensions of γ/s/cm³. The adjoint equation corresponding to Eq. (1) is

$$-\frac{d\psi^{*g}(r)}{dr} + \Sigma_t^g(r)\psi^{*g}(r) = 0, \qquad \psi^{*g}(r_d') = S^{*g}, \quad g = 1,\ldots,G. \tag{2}$$

Suppose the location of the "outer" boundary of the source, r_1', is unknown and to be determined.

The solution of Eq. (1) for the unknown system, in which r_1 represents an assumed value for the exact value r_1', is

$$\psi^g(r) = \begin{cases} \dfrac{q^g}{\Sigma_t^g}(1 - e^{-\Sigma_t^g r}), & 0 \le r < r_1 \\ M^g, & r_1 \le r \le r_d', \end{cases} \tag{3}$$

where M^g is the calculated leakage:

$$M^g = \psi^g(r_d') = \frac{q^g}{\Sigma_t^g}(1 - e^{-\Sigma_t^g r_1}). \tag{4}$$

The leakage that would be "measured" for the actual system is

$$M_o^g = \psi_o^g(r_d') = \frac{q^g}{\Sigma_t^g}(1 - e^{-\Sigma_t^g r_1'}), \tag{5}$$

where subscript o represents the solution for the actual system.

The solution of Eq. (2) is

$$\psi^{*g}(r) = \begin{cases} S^{*g} e^{-\Sigma_t^g (r_1 - r)}, & 0 \le r < r_1 \\ S^{*g}, & r_1 \le r \le r_d'. \end{cases} \tag{6}$$

APPLICATION OF THE SCHWINGER METHOD

The Schwinger inverse method [1] was derived from a perturbation-theory approach to the inverse transport problem. Instead of calculating the effect of a system perturbation on a quantity of interest (the usual use of the Schwinger functional), the quantity of interest was assumed to be given (from a measurement) and the Schwinger functional was manipulated to produce an equation for the system perturbation. The equation is applied iteratively.

Specializing to the simplified case of this paper, the basic equation for the Schwinger inverse method of finding an unknown source interface [1] becomes

$$\frac{\Delta r_1}{\int dV \psi^{*g} q^g}\left[\Sigma_t^g \left.\frac{\partial I^g(r)}{\partial r}\right|_{r=r_1} - \frac{M^g}{M_o^g} q^g \left.\frac{\partial I_s^{*g}(r)}{\partial r}\right|_{r=r_1}\right] = \frac{M^g - M_o^g}{M_o^g}, \quad g = 1, \ldots, G, \tag{7}$$

where Δr_1 is the update, calculated in the present iteration, to the present estimate r_1 and, in the monodirectional problem, $I^g(r) \equiv \int_{r_1}^{r} dr \psi^{*g} \psi^g$ and $I_s^{*g}(r) \equiv \int_{r_1}^{r} dr \psi^{*g}$. The derivatives $\partial I^g(r)/\partial r\big|_{r=r_1}$ and $\partial I_s^{*g}(r)/\partial r\big|_{r=r_1}$ appear because of a first-order Taylor expansion of $I^g(r)$ and $I_s^{*g}(r)$, respectively, in the derivation of Eq. (7) [1].

In the Schwinger inverse method for this problem, $S^{*g} = 1$. Using this fact and Eq. (6), the integral outside the brackets in Eq. (7) is

$$\begin{aligned}\int dV \psi^{*g} q^g &= \int_0^{r_1} dr e^{-\Sigma_t^g (r_1 - r)} q^g \\ &= \frac{q^g}{\Sigma_t^g}\left(1 - e^{-\Sigma_t^g r_1}\right) = M^g,\end{aligned} \tag{8}$$

a well-known result obtainable from the definition of the adjoint transport operator [1].

The quantity $I^g(r)$ is, in the monodirectional problem,

$$\begin{aligned} I^g(r) &= \begin{cases} \int_{r_1}^{r} dr_0 e^{-\Sigma_t^g (r_1 - r_0)} \frac{q^g}{\Sigma_t^g}(1 - e^{-\Sigma_t^g r_0}), & r < r_1 \\ \int_{r_1}^{r} dr_0 M^g, & r > r_1 \end{cases} \\ &= \begin{cases} \frac{q^g}{\Sigma_t^g}\left(\frac{1}{\Sigma_t^g} e^{-\Sigma_t^g (r_1 - r)} - \frac{1}{\Sigma_t^g} + (r_1 - r) e^{-\Sigma_t^g r_1}\right), & r < r_1 \\ M^g (r - r_1), & r > r_1. \end{cases}\end{aligned} \tag{9}$$

The derivative is

$$\left.\frac{\partial I^g(r)}{\partial r}\right|_{r=r_1} = \begin{cases} \frac{q^g}{\Sigma_t^g}\left.\left(e^{-\Sigma_t^g (r_1 - r)} - e^{-\Sigma_t^g r_1}\right)\right|_{r=r_1}, & r < r_1 \\ M^g, & r > r_1 \end{cases} = M^g. \tag{10}$$

The quantity $I_s^{*g}(r)$ is, in the monodirectional problem,

$$\begin{aligned} I_s^{*g}(r) &= \begin{cases} \int_{r_1}^{r} dr_0 e^{-\Sigma_t^g (r_1 - r_0)}, & r < r_1 \\ \int_{r_1}^{r} dr_0, & r > r_1 \end{cases} \\ &= \begin{cases} \frac{1}{\Sigma_t^g}\left(e^{-\Sigma_t^g (r_1 - r)} - 1\right), & r < r_1 \\ r - r_1, & r > r_1. \end{cases}\end{aligned} \tag{11}$$

The derivative is

$$\left.\frac{\partial I_s^{*g}(r)}{\partial r}\right|_{r=r_1} = \begin{cases} \left. e^{-\Sigma_t^g (r_1 - r)} \right|_{r=r_1}, & r < r_1 \\ 1, & r > r_1 \end{cases} = 1. \tag{12}$$

Note that the first-order Taylor expansion that leads to the inclusion of $\partial I^g(r)/\partial r\big|_{r=r_1}$ and $\partial I_s^{*g}(r)/\partial r\big|_{r=r_1}$ in Eq. (7) is a perfectly good approximation for $I^g(r)$ [Eq. (9)] and $I_s^{*g}(r)$ [Eq. (11)] on the right side of the interface ($r > r_1$), but a potentially poor approximation for the left side of the interface ($r < r_1$), depending on the magnitude of Σ_t^g. This observation helps explain the method's failure to converge [1] in certain problems when the source radius was unknown, and suggests that inclusion of more terms in the Taylor series might be useful, although the resulting equation, unlike Eq. (7), would be nonlinear in Δr_1.

Specializing to the case of a single energy group ($G = 1$) and using Eqs. (8), (10), and (12), Eq. (7) becomes

$$\frac{\Delta r_1}{M^1}\left[\Sigma_t^1 M^1 - \frac{M^1}{M_o^1} q^1\right] = \frac{M^1}{M_o^1} - 1. \tag{13}$$

Using Eqs. (4) and (5) and a bit of algebra, Eq. (13) becomes

$$\Delta r_1 = \frac{1}{\Sigma_t^1}\left(e^{\Sigma_t^1 \Delta r_{1,ex}} - 1\right), \tag{14}$$

where $\Delta r_{1,ex} \equiv r_1' - r_1$ is the difference between the exact interface location and the current model estimate.

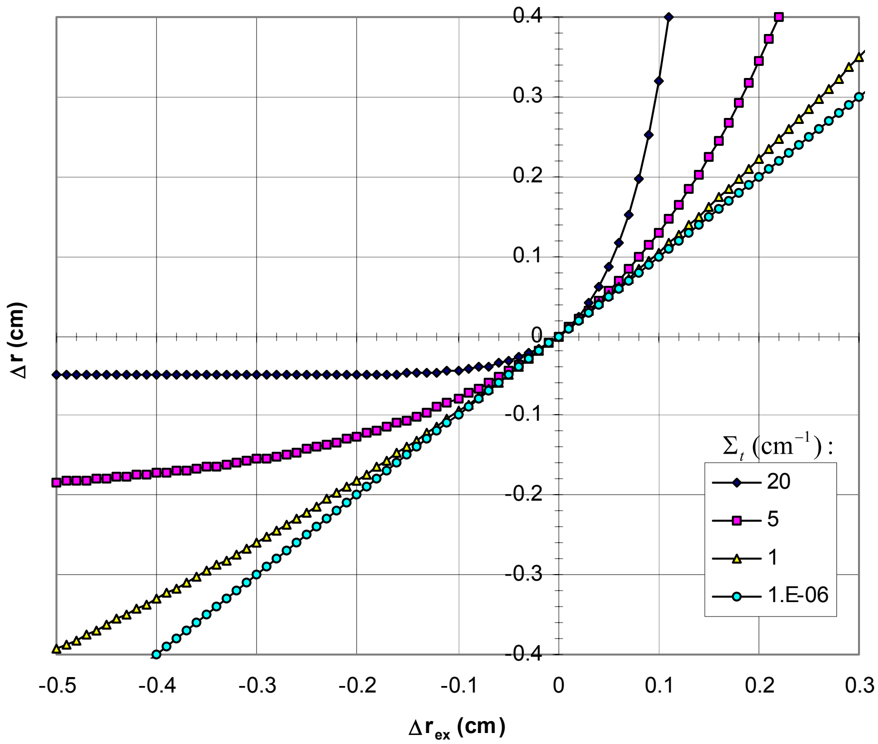

Fig. 1. Convergence of the Schwinger inverse method for a homogeneous source with one unknown boundary [plots of Eq. (14)].

ANALYTIC RESULTS

The convergence of the Schwinger method for the monodirectional one-region source problem with an unknown right-hand boundary location is shown in Fig. 1 for various values of Σ_t^1. It is evident that, for this problem, the method will always converge, regardless of the starting value of r_1 or the accuracy of the first-order Taylor expansion for the integrals.

On the other hand, when Σ_t^1 is large, the method may have difficulty converging, and it may even be forced into nonphysical models before convergence. For example, if $\Sigma_t^1 = 20\ \text{cm}^{-1}$ and the exact location is $r_1' = 10$ cm and the initial guess is $r_1 = 9.5$ cm, implying $\Delta r_{1,ex} = 0.5$ cm, then Eq. (14) yields $\Delta r_1 = 1101.3$ cm and the value for the next iteration is $r_1 = r_1 + \Delta r_1 = 1110.8$ cm, which is probably a nonphysical value if the expected value is close to 9.5 cm. Because Eq. (14) is unconstrained by the need for a physical model, however, this problem actually converges, creeping along the asymptote $\Delta r_1 = -0.05$ cm for 22 000 iterations.

CONCLUSIONS

Analytic solutions to a simple problem have revealed reasons for convergence trouble in the Schwinger inverse method [1]. Multilayer slab problems with multiple unknown interface locations (both source and shield) will be presented in the talk.

REFERENCES

1. J. A. FAVORITE, “Using the Schwinger Variational Functional for the Solution of Inverse Transport Problems,” *Nucl. Sci. Eng.*, **146**, to appear (2004).